Real Complexity Theory: from Computability to Implementations, from Heuristics to Rigorous & Optimal Algorithms

Martin Ziegler
The purpose of computing is insight, not numbers.

Richard Wesley Hamming (1962)

Is there anything a numerical engineer cannot compute?

nag_opt_one_var_deriv (e04bbc) normally computes a sequence of $x$ values which tend in the limit to a minimum of $F(x)$ subject to the given bounds.
"How do engineers deal with the problem of assigning some measure of reliability to the numbers that the computer produces? Over the years, I have sat on many Ph.D. qualifying examinations or dissertation defenses for engineering students whose work involved a significant amount of numerical computing. In one form or another, I invariably ask two questions "Why did you choose that particular algorithm?" and "How do you know that your answers are as accurate as you claim?". The first question is usually answered confidently, using such terms as "second-order convergence" or "von Neumann stability criterion". The next question, alas, tends to be embarrassing. After an initial blank or hostile stare, I usually get an answer like "I tested the method with some simple examples and it worked", "I repeated the computation with several values of n and the results agreed to three decimal places", or more lamely, "the answers looked like what I expected". So far, I have not faulted any student for the unsatisfactory nature of such a response. One reason for my reluctance to criticize is that I have really nothing better to offer. Rigorous analysis is out of the question."}

### Parameterized Uniform Real Complexity Theory

**Origin**
- **Logic + Real Analysis**
- **Computer Science + Numerical Analysis**

**Example results**
- There exists a computable smooth \( f \): \([0;1]^2 \rightarrow [0;1]^2\) with no computable fixed point.
- The Mandelbrot Set is computable, subject to the hyperbolicity conjecture.

**Type of Problems; Specification**
- Computability of certain real numbers, multi-/ functions etc., formalized and precisely.

**Methods**
- Prove existence / impossibility of algorithm solving some problem.

**Intermediate precision**
- Unbounded

**Programming**
- Type-2 Machine / real PCF

**Consistent semantics?**
- Yes, closed under composition

**Notion of Efficiency**
- (recursive enumerability)

**Correctness/Optimality Demonstration**
- Formal proof, derived from specification

**Establishing optimality**
- Embedding undecidable (e.g. halting) problems, Weihrauch-reduction

### High-Precision/Validated Numerics

- Algebraic & transcend. number theory, Dynamical Systems
- Certain real constants: formal and precise
- GMP + individually tailored software

### Numerical Engineering

- Engineering + Numerics
- (individual) design-/parameters, informal
- Heuristical recipes and experience
- Fixed: hardware float/double
- NAG, MATLAB

### Complexity of Real Analysis + Computational Complexity + Logic + IBC

- Formal parameters + optimal algorithms computing real functions & operators
- Natural parameters + optimal algorithms computing real functions & operators
- Modify new & rigorously analyze existing algorithms
- Certain real constants: formal and precise
- Case-by-case
- Linear growth
- GMP + individually tailored software
- Custom / ad-hoc

### Complexity of real functions and subsets; formal and precise

- Polynomial-time computable s.t. solution \( z: [0;1] \rightarrow [-1;1] \) of ODE \( \dot{z}(t) = f(z(t), t) \) is not polynomial-time computable unless \( P = \text{PSPACE} \).
- For polynomal analytic function \( f \), \( z \) is polytime.

### Complexity of real Analysis + Computational Complexity + Logic + IBC

- Computational Benefit of Smoothness: A Rigorous Parameterized Complexity Analysis of Operators on Gevrey’s Hierarchy
- Uniform Complexity of Operators on Compact Sets
- The Lorenz Attractor exists.
- The Kepler Conjecture is true
- Billions of digits of \( \pi \)
- Numerical verification of the Riemann Hypothesis
Benefits of Complexity Theory over Continuous Universes

- Full specification (input/output behavior) \(i\text{RRAM}\)
- of problems over real numbers, functions, sets
- Consistent semantics (e.g. tests) closed under composition for modular approach to software
- Canonical interface declarations (TTE+2\(^{nd}\) ord.)
- Rigorous convergence & runtime analyses
- turning recipes and heuristics into algorithms
- Realistic performance guarantees in bit model
- and optimality proofs from IBC or relative to common conjectures in classical complexity
- and parametrized for fine-grained predictions
Computable Real Numbers

Theorem: For $r \in \mathbb{R}$, the following are equivalent:

a) $r$ has some/all binary expansion decidable

b) There is an algorithm printing, on input $m \in \mathbb{N}$, some $a \in \mathbb{Z}$ with $|r - a/2^m| \leq 2^{-m}$

c) There is an algorithm printing two sequences $(q_n) \subseteq \mathbb{Q}$ and $(\varepsilon_n)$ with $|r - q_n| \leq \varepsilon_n \rightarrow 0$

d) There is an algorithm printing $(q_n) \subseteq \mathbb{Q}$ with $q_n \rightarrow r$. H := \{ \langle B, x \rangle : \text{algorithm } B \text{ terminates on input } x \} \subseteq \mathbb{N}
Real Function Complexity

Function $f: [0,1] \rightarrow \mathbb{R}$ computable if some TM can, on input of $(a_m) \subseteq \mathbb{Z}$ with $|x-a_m/2^m| < 2^{-m}$ in time $t(n)$ output $b \in \mathbb{Z}$ with

$$|f(x) - b| < 2^{-\mu(n)} \Rightarrow |f(x) - f(y)| < 2^{-n}$$

Examples:

a) $+ , \times , \exp$ polytime on $[0;1]!$

b) $f(x) \equiv \sum_{n \in L} 4^{-n}$ iff $L \subseteq \{0,1\}^*$ polytime-decidable

c) $\text{sign}(\cdot)$ not polytime-computable
   Lipschitz, Hölder

Observation

i) $f$ computable $\Rightarrow$ continuous.

ii) If $f$ is computable in time $t(n)$, then $t(n)$ is a modulus of uniform continuity of $f$. 
Three Effects in Real Computation
that numerical scientists might be interested in / should be aware of

a) natural emergence of **multivaluedness** 
   (aka non-extensionality) \( \rightarrow \) semantics of "\( <_n \)"

b) Uniform computation may require **discrete advice**

**Example:** Given \( x \in \mathbb{R} \), compute \( n \in \mathbb{Z} \) with \( n \geq x \).

**Example (E. Specker'69):** Given \( a_0, \ldots, a_{d-1} \in \mathbb{C} \)
compute a \( d \)-tuple \( z_1, \ldots, z_d \in \mathbb{C} \) of roots of
\[
p(z) := a_0 + z \cdot d_1 + \ldots + z^{d-1} \cdot a_{d-1} + z^d
\]
with multiplicities.

Finding an eigenvector basis to a given real

**Example:** +, \( \exp \) computable in time polynomial in \( n \) on \([0;1] \); on \([0;2^k] \): + in time polynomial in \( n+k \), \( \exp \) in time polynomial in \( n+2^k \).
Recap on Structural Complexity

- \( L \subseteq \{0,1\}^* \) **polynomial-time decidable** if a Turing Machine can, given \( v \in \{0,1\}^n \), tell whether \( v \in L \) or \( v \notin L \) within a number of steps polynomial in \( n = |v| \).

- \( f : \{0,1\}^* \rightarrow \{0,1\}^* \) **polynomial-time computable** if...

- \( L \) is **polynomial-space decidable** if can tell whether \( v \in L \) or \( v \notin L \) using at most polynomially many memory bits.

- \( L = \{ v \in \{1\}^* : \exists w \in \{0,1\}^{p(|x|)} : \langle v,w \rangle \in V \} \) with \( p \in \mathbb{N}[\mathbb{N}] \) and \( V \in \mathcal{P} \) is called **polynomial-time verifiable**.

- \( f \in \mathcal{FP} \) is a **polynomial-time reduction** from \( A \) to \( B \) ("\( A \leq^p B \)") if it holds: \( \forall v : v \in A \iff f(v) \in B \). \( B \in \mathcal{NP} \) is **\( \mathcal{NP} \)-complete** if \( A \leq^p B \) for every \( A \in \mathcal{NP} \).
Nonuniform Complexity of Operators

\[ f : [0;1] \rightarrow [0;1] \] polytime computable \( \implies \) continuous

- Max: \( f \rightarrow \text{Max}(f) : x \rightarrow \max\{ f(t) : t \leq x \} \)
  
  \( \text{Max}(f) \) computable in exp. time;
  
  polytime-computable \iff \( P = NP \)

\[ \{ \langle \text{bin}(a), \text{bin}(b), 2^n \rangle : \exists w \leq a \leq 2^m, f(w/2^m) > b/2^n, m := \mu(n) \} \]

- \( \int : f \rightarrow \int f : (x \rightarrow \int_0^x f(t) \, dt) \)
  
  \( \mu \) polyn. modulus of continuity

Finding starting points for Newton Iteration is hard

\( F_\infty = \#P \)

\( F_1 = \#P_1 \)

\[ \frac{\partial}{\partial t} \rightarrow \bar{z}: \quad \frac{\partial}{\partial t} = f(t, \bar{z}) \]

Initial value for solution \( \bar{z}(0) \)

\( \mu \) polyn. modulus of continuity

\[ \#P \]

in general no computable solution

\( [Friedman\&Ko'82ff] \)

[Kawamura et al]
'Max is $\mathcal{NP}$-hard'

$$\mathcal{NP} \ni L = \{ N \in \mathbb{N} \mid \exists M < N. \langle N, M \rangle \in V \}, \forall V \in \mathcal{P}$$

$$C^\infty g_L : t \rightarrow \sum_{\langle N, M \rangle \in V} \phi(3tN^3 - 3N^2/M) / N$$

$\phi(t) = \exp(-t^2/1-t^2)$

$C^\infty$ 'pulse' function

polytime computable

To every $L \in \mathcal{NP}$ there exists a polytime computable $C^\infty$ function $g_L : [0,1] \rightarrow \mathbb{R}$ s.t.:

$[0,1] \ni t \rightarrow \max g_L|_{[0,t]}$ again polytime iff $L \in \mathcal{P}$
Complexity of PDEs

- [Pour-El&Richards'81, ..., Weihrauch&Zhong'02] In-/computability of the Wave Equation (hyperbolic)
- Computability of some non-/linear PDEs: [Weihrauch&Zhong'99ff, Brattka&Yoshikawa'06, ...]

Poisson's Equation:
\[ \Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial \Omega \]

- electrostatic / gravitational potential of the charge/mass distribution \( f \) with boundary condition \( g \)
- 2nd order, linear, elliptic: homog. \((f,0)\) and inh. \((0,g)\)
- 'fundamental' solutions \( \ln |x| \) (2D) and \( 1/|x| \) (3D)
- 'explicit' Green's functions for various domains,
- solution formula on the complex unit disc: e.g., \( u(z) = -\frac{1}{2\pi} \int_1^{\infty} \frac{\ln|w-z|}{|w\cdot z^*-1|} \cdot f(w) \, dA(w) \)
Green's Function in 2D

\[ G(1, w) \]

\[ \Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega \]

\[ G(z, w) = 0, 0 \]

\[ \ln \frac{|w - z|}{|w \cdot z^* - 1|} \]
**Theorem:** $B_d := \text{closed} \ d$-dim. Euklidean unit ball

a) For every polytime $f: B_d \rightarrow \mathbb{R}$ and $g: \partial B_d \rightarrow \mathbb{R}$, there exists a unique $C^2$ solution $u$ and $u$ is computable in exponential time.

b) If $FP = \#P$, then $u$ is even polytime computable.

c) There exists a polytime $f \in C^\infty$ such that $u$ to $g \equiv 0$ is polytime iff $FP = \#P$.

d) For $d > 1$ there is a polytime $g \in C^\infty$ s.t. $u$ to $f \equiv 0$ is polytime iff $FP = \#P$.

\[
\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial \Omega
\]
Complexity of Operators (cont.d)

\( f: [0;1] \to [0;1] \) polytime computable + \( C^\omega \) (analytic)

- **Max**: \( f \to \text{Max}(f): x \to \max \{ f(t): t \leq x \} \)
  polytime-computable!

- \( \int : f \to \int f: (x \to \int_0^x f(t) \, dt) \)
  polytime-computable!

- **dsolve**: \( C[0;1] \times [-1;1] \ni f \to z: \dot{z}(t) = f(t,z), \ z(0) = 0. \)
  polytime-computable!

**Claims are non-uniform**: parameterized

- Runtime (polyn. degree) of computing \( \text{Max}(f) \) in dependence on runtime (polyn. degree) of \( f \)?
- Discrete advice about (in addition to apprx. to) \( f \) employed by computation of \( \text{Max}(f) \)?

[ N.Müller'87+'95, Müller&Moiske'93, Bournez et al.'11ff ]
Representing Power Series on the closed unit disc

\[ \sum_j c_j z^j \]

- radius of convergence: \( R = \frac{1}{\limsup_j |c_j|^{1/j}} \)
- to \( 0 < r < R \) exist \( k \in \mathbb{N} : |c_j| \leq 2^k / r^j \) (Cauchy-Hadamard)
- \( \mathbb{N} \ni k \geq 1 / \log(r) = \Theta(1/(r-1)) \)
- tail bound: \( |\sum_{j \geq N} c_j z^j| \leq C \cdot (|z|/r)^N / (1-|z|/r) \)

**Complexity uniform in** \( |z| \leq 1 \): (i.e. \( R > 1 \))

Convergence degrades as \( r \rightarrow 1 \); quantitatively?

**Theorem:**
Represent series \( \sum_j c_j z^j \) with \( R > 1 \) as
\( (a_{jm}) \subseteq \mathbb{Z} \) s.t. \( |c_j - a_{jm}/2^m| < 2^{-m} \) \( (\equiv \rho^\omega) \) and \( k \in \mathbb{N} \) as above

The following are uniformly computable in time, parametrized running time, output precision:

- i) eval, ii) sum, iii) product, iv) derivative, v) anti-derivative \( \int \), vi) Max

[Kawamura, Müller, Rösnick, Z.'14]
Representing Continuous Functions

**TTE:** Represent space $\mathbb{Z}$ via surj. $\delta: \{0,1\}^\omega \rightarrow \mathbb{Z}$

On compact $\mathcal{Lip}_1 := \{ f:[0;1] \rightarrow [0;1] \text{ 1-Lipschitz} \}$, representing $f$ via a $\rho^\omega$-name of $(f(k/2^n))_{(k, n)}$ renders operator $\text{Apply}: \mathcal{Lip}_1 \times [0;1] \ni (f, x) \rightarrow f(x)$ computable in time $O(1)^n$, not in subexp. time for any representat.

$\geq 2^{2^{n-1}}$ functions pairwise differing when evaluating up to error $2^{-n}$ but only $2^{t(n)}$ different initial segments of $\delta$-names that can be read within $t(n)$ steps. q.e.d.

$\mathcal{Lip}_1$ has exponential metric entropy [Weihrauch'03]

IBC: exact (or of fixed precision) *unit cost* queries

Friedman&Ko: oracle access to $f$ via approx. eval.:

Given $q \in \mathcal{D}_n := \{a/2^n: a \in \mathbb{Z} \}$ return $p \in \mathcal{D}_n: |f(q) - p| < 2^{-n}$. 
Representing Continuous Functions

**TTE:** Represent space $\mathbb{Z}$ via surj. $\delta : \{0, 1\}^\omega \to \mathbb{Z}$

**Definition [Kawamura&Cook'10]:**
A $2^{\text{nd}}$-order representation is a surj. $\delta : \{0, 1\}^{**} \to \mathbb{Z}$ where $\{0, 1\}^{**} = (\{0, 1\}^*)^* \{0, 1\}^*$ denotes Baire Space [Kawamura&Pauly'14]

**Example:** Define a $\rho^\mathbb{D}$-name of $f \in C[0;1]$ as any $\phi$ s.t.

$$|f(\text{bin}(v)/2^{|v|}) - \text{bin}(\phi(v))/2^{|v|}| < 2^{-|v|}$$

Friedman&Ko: oracle access to $f$ via approx. eval.:
Given $q \in \mathbb{D}_n := \{a/2^n : a \in \mathbb{Z}\}$ return $p \in \mathbb{D}_n : |f(q) - p| < 2^{-n}$. 

Oracle machine computing a functional $\Lambda : C[0;1] \times \mathbb{R} \to \mathbb{R}$
Representing Real Analytic Functions

**Definition:** \( C^\omega[-1,1] := \{ f: [-1;1] \to \mathbb{R} \text{ restriction of complex differentiable } g: U \to \mathbb{C}, [-1;1] \subseteq U \subseteq \mathbb{C} \text{ open} \} \)

**Fact:** \( f \in C^\omega[-1;1] \iff f \in C^\infty \land \exists k \in \mathbb{N} \land \forall j: |f^{(j)}| \leq 2^k \cdot k^j \cdot j! \)

\( \iff f = \text{finitely many local power series on } [-1;1] \)

**Theorem:** The following (2nd-ord) representations of \( C^\omega[-1,1] \) are parameterized poly-time equivalent:

a) a \( \rho^D \)-name of \( f \) with advice=param. \( k \) as above,

b) a \( \rho^D \)-name of \( f \) with \( \ell \)

s.t. \( R_\ell \subseteq U \) and \( |g| \leq 2^\ell \) on \( R_\ell \)

c) \( \rho^\omega \)-names of \( m \) power series \( (c_j^{(1)}), ..., (c_j^{(m)}) \subseteq \mathbb{R} \) of \( f \) around equidistant centers \( \in [-1;1] \) s.t. \( |c_j| \leq 2^m \cdot (2^m)^j \)

**Theorem:** On \( C^\omega[0,1], \) i) eval ii) sum ... vi) max are computable with parameterized polynomial time
Gevrey's Function Hierarchy

**Definition** (Maurice Gevrey 1918, studying PDEs)

\[ g \in G_k^\ell [-1;1] \iff \forall j: \|g^{(j)}\| \leq 2^k \cdot k^j \cdot j! \]

**Fact:** \( f \in C^\omega[-1;1] \iff f \in C^\infty \land \exists k \in \mathbb{N} \ \forall j: \|f^{(j)}\| \leq 2^k \cdot k^j \cdot j! \)

\[ G^\ell := \bigcup_k G_k^\ell \]

\[ G^1 = C^\omega \]

**Example:** The following \( g \) is not analytic but in \( G^3[-1;1] \)

\[ g(x) := \exp \left( \frac{x^2}{x^2 - 1} \right) \text{ for } |x| \leq 1, \]

\[ g(x) := 0 \text{ for } |x| \geq 1 \]
Komplexity on Gevrey's Hierarchy

**Definition** (Maurice Gevrey 1918, studying PDEs)

\( g \in G^\ell_k[-1;1] :\Longleftrightarrow \forall j: \|g^{(j)}\| \leq 2^k \cdot k^j \cdot j^{\cdot \ell} \)

\( G^\ell := \bigcup_k G^\ell_k \)

**Labhalla&Lombardi&Moutai 2001:**

\[ \Rightarrow \exists B \ \forall n \ \exists p \in \mathbb{D}[X]: \deg(p) < B \cdot n^\ell \ \|g-p\| \leq 2^{-n} \Rightarrow g \in G^{2\ell-1}[-1;1] \]

**Theorem [Kawamura,Müller,Rösnick,Z.'14]:**

The following (2\textsuperscript{nd}-ord) representations of \( G[-1;1] := \bigcup_\ell G^\ell[-1;1] \) are computationally equivalent up to time polynomial in \((k+n)^\ell\):

a) \( \rho^\mathcal{D} \)-name of \( f \) with advice \( k, \ell \) as above
b) sequence \( p_n \in \mathbb{D}[X] \) with \( \deg(p_n) < B \cdot n^\ell \ \|g-p_n\|_\infty \leq 2^{-n} \).

Moreover they render the following operations computable in time polynomial in \((k+n)^\ell\):

i) eval, ii) sum, iii) product, iv) \( d/dx \), v) \( \int \), vi) max W.r.t. \( \rho^\mathcal{D} \), max (vi) on \( G^\ell_1 \) requires time \( \Omega(n^\ell) \).
Representing Compact Euclidean Subsets

Fix closed $S \subseteq [0;1]^d$ and $1 \leq p \leq \infty$.

$\psi_p$-name $\varphi$: $\varphi(q)=1$ for $q \in D_n^d$ with $\text{Ball}_p(q,2^{-n}) \cap S \neq \emptyset$

$\varphi(q)=0$ for $q \in D_n^d$ with $\text{Ball}_p(q,2^{-n+1}) \cap S = \emptyset$

$\delta_p$-name: $\rho^D$-name of $\text{dist}_{S,p} : [0;1]^d \ni x \rightarrow \min \{ |x-s|_p : s \in S \}$

$\omega_p$-name $\varphi$: $\varphi(q)=1$ for $q \in D_n^d$ with $\text{Ball}_p(q,2^{-n}) \subseteq S$

$\varphi(q)=0$ for $q \in D_n^d$ with $\text{Ball}_p(q,2^{-n}) \cap S = \emptyset$

(symm./rel. distance, multiv. best approx...)

A $2^{\text{nd}}$-order representation of $Z$ is a surjective $\delta : \subseteq(\{0,1\}^*)^*(\{0,1\})^* \rightarrow Z$

[Brattka&Weihrauch'99,Z.'02+'04, Braverman'04, Zhao&Müller'08, Rösnick'14]
Representing Compact Euclidean Subsets

Fix closed $S \subseteq [0;1]^d$ and $1 \leq p \leq \infty$.

$\psi_p$-name $\varphi$: $\varphi(q) = 1$ for $q \in D_n^d$ with $\text{Ball}_p(q, 2^{-n}) \cap S \neq \emptyset$

$\varphi(q) = 0$ for $q \in D_n^d$ with $\text{Ball}_p(q, 2^{-n+1}) \cap S = \emptyset$

$\delta_p$-name: $\rho^d$-name of $\text{dist}_{S,p} : [0;1]^d \ni x \rightarrow \min \{|x-s|_p : s \in S\}$

$\omega_p$-name

**Fact**: $\psi \equiv \delta$, $\equiv \omega$ on regular sets [Bra&Wei'99,Z.'02]

**Theorem** [Brav.'04,Zh&Mü'08,Rösn.'14]:

a) It holds $\delta \leq^P \psi \leq^P \omega$. Projection is $NP$-"complete"

b) There is a $S$ with $\text{dist}_{S,1}$ polytime but not $\text{dist}_{S,\infty}$

c) For any (!) fixed $1 \leq p, p' \leq \infty$ it holds $\psi_p \equiv^P \psi_{p'}$

d) For convex sets of inner diameter $\geq 2^{-k}$, all three representations parametriz. polyn.time equivalent.

unless $P = NP$
Function arguments $f \in C[0;1]$: 2nd-order represent. Apply: $Lip_{2^\epsilon}[0;1] \times [0;1] \ni (f,x) \to f(x)$ computable in parameterized time polynomial in $n + \ell$.

**Observation:** If $f$ is computable in time $t(n)$, then $t(n)$ is a modulus of uniform continuity of $f$.

Apply: $C[0;1] \times [0;1] \ni (f,x) \to f(x)$ requires time depending on a modulus $\mu$ of continuity: "Parameter" $\mu$ is not $\mathbb{N}$-valued but $\mathbb{N}^\mathbb{N}$-valued!

**Example:** $\lambda^3(\lambda(n^2) \cdot n + \lambda^2(n)) + n^{17}$

**Def [Mehlhorn'76]:** A 2nd-order polynomial $P(n,\lambda)$ is a term over $\oplus, \times$, integer constants and (1st-order) variable $n$ (ranging over $\mathbb{N}$) and 2nd-order variable $\lambda$ (ranging over $\mathbb{N}^\mathbb{N}$).
2nd-Order Polynomials

Observation: a) 2nd-order polynomials are closed under both kinds of composition

\[(Q \circ P)(n,\lambda) := Q(P(n,\lambda),\lambda) \quad \text{and} \quad (Q \circ P)(n,\lambda) := Q(n,P(\cdot,\lambda))\]

b) For \(\lambda \in \mathbb{N}[n]\), \(P(n,\lambda)\) is an ordinary polynomial.

Example: \(\lambda^3(\lambda(n^2) \cdot n + \lambda^2(n)) + n^{17}\)

Def [Mehlhorn'76]: A 2nd-order polynomial \(P(n,\lambda)\) is a term over +, \(\times\), integer constants and (1st-order) variable \(n\) (ranging over \(\mathbb{N}\)) and 2nd-order variable \(\lambda\) (ranging over \(\mathbb{N}^\mathbb{N}\)).
2nd-Order Polyn.time Computation

A 2nd-order representation is a surj. \( \delta: \subseteq \{0,1\}^{**} \rightarrow \mathbb{Z} \) with Baire Space \( \{0,1\}^{**} = (\{0,1\}^*)^{(\{0,1\})^*} \) and grading \( |\varphi(n)| = \max\{|\varphi(\nu)| : |\nu| \leq n\} \).

A \( \rho^D \)-name of any \( \varphi \) s.t. \( \left| f\left( \frac{\text{bin}(\nu)}{2^{|\nu|}} \right) - \frac{\text{bin}(\varphi(\nu))}{2^{|\nu|}} \right| < 2^{-|\nu|} \)

\( \xi \)-name of \( \nu \xrightarrow{\varphi(\nu)} \xrightarrow{\nu} \varphi(\nu) \in \mathbb{Z} \)

Oracle machine computing a functional \( \Lambda: \subseteq \mathbb{Z} \times X \rightarrow \mathbb{R} \)

Def [Kapron&Cook'96]: \( \mathcal{M} \) runs in 2nd-order poly. time if \#steps \( \leq P(n,|\varphi|) \) for every \( x \) and \( \xi \)-name \( \varphi \).

Def [Mehlhorn'76]: A 2nd-order polynomial \( P(n,\lambda) \) is a term over +, \times, integer constants and (1st-order) variable \( n \) (ranging over \( \mathbb{N} \)) and 2nd-order variable \( \lambda \) (ranging over \( \mathbb{N}^{\mathbb{N}} \)).
Real Complexity Theory: Foundation for the Future of Mathematical Numerics

- Full specification (input/output behavior) \textit{iRRAM}
- of problems over real numbers, functions, sets
- Consistent semantics (e.g. tests) closed under composition for modular approach to software
- Canonical interface declarations (TTE+2\textsuperscript{nd} ord.)
- Rigorous convergence & runtime analyses
- turning recipes and heuristics into algorithms
- Realistic performance guarantees in bit model
- and optimality proofs from IBC or relative to common conjectures in classical complexity
- and parametrized for fine-grained predictions