TECHNISCHE UNIVERSITÄT DARMSTADT

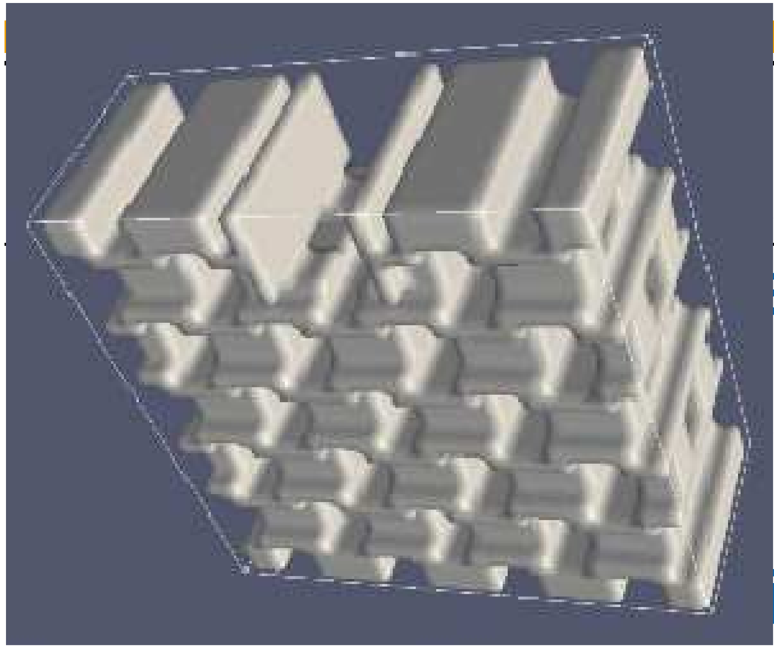
Mathematik

Real Complexity Theory:

from Computability to Implementations, from Heuristics to Rigorous & Optimal Algorithms

TECHNISCHE UNIVERSITÄT DARMSTADT

Martin Ziegler



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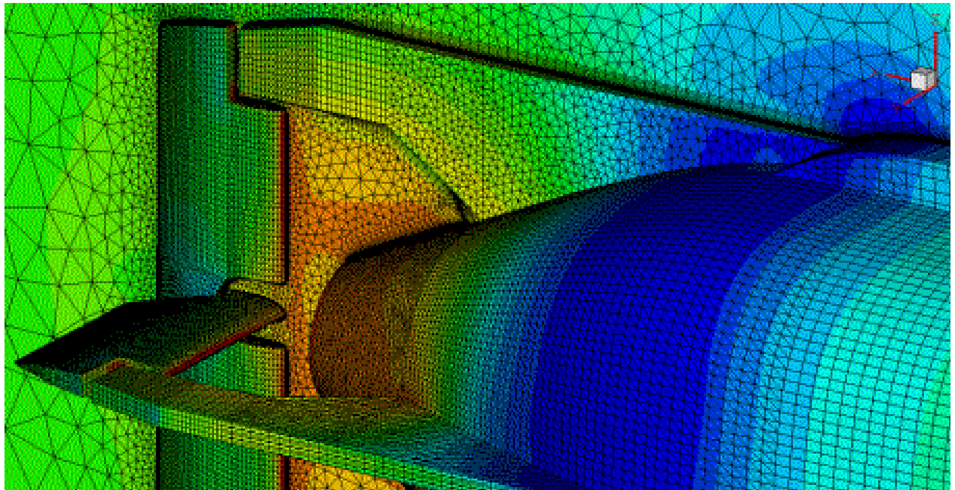
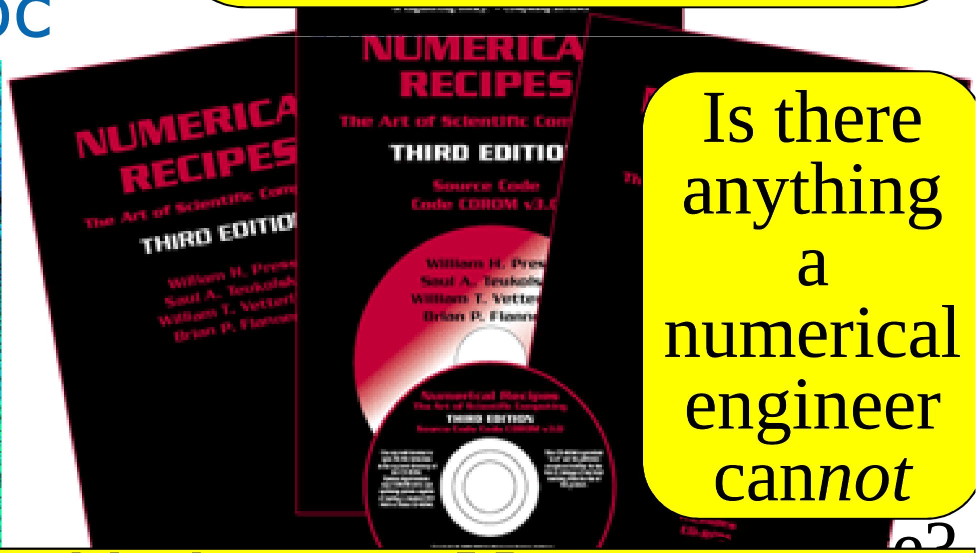
The purpose of computing is insight, not numbers.
Richard Wesley Hamming (1962)

- instance-based / ad-hoc





Is there anything a numerical engineer can*not*

nag_opt_one_var_deriv (e04bbc) normally computes a sequence of x values which tend in the limit to a minimum of F x subject to the given bounds

Rigour in Numerics?

«How do engineers deal with the problem of assigning some measure of reliability to the numbers that the computer produces? Over the years, I have sat on many Ph.D. qualifying examinations or dissertation defenses for engineering students whose work involved a significant amount of numerical computing. In one form or another, I invariably ask two questions "Why did you choose that particular algorithm?" and "How do you know that your answers are as accurate as you claim?". *The first question is usually answered confidently, using such terms as "second-order convergence" or "von Neumann stability criterion". The next question, alas, tends to be embarrassing. After an initial blank or hostile stare, I usually get an answer like* "I tested the method with some simple examples and it worked", "I repeated the computation with several values of n and the results agreed to three decimal places", *or more lamely,* "the answers looked like what I expected". *So far, I have not faulted any student for the unsatisfactory nature of such a response. One reason for my reluctance to criticize is that I have really nothing better to offer. <u>Rigorous analysis is out of the question</u>.*» Peter Linz, p.412, Bull. AMS vol.19:2 (1988).

Parameterized Uniform Real Complexity Theory

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	Recursive Analysis	Non-Uniform Real Complexity	Yes, we can! (bridge the gap)	High-Precision/ Validated Numerics	Numerical Engineering
Origin	Logic + Real Analysis	Computer Science + Numerical Analysis	Analysis + Computational Complexity + Logic + IBC	Algebraic & transcend. number theory, Dynamical Systems	Engineering + Numerics
Example results	• There exists a computable smooth $f:[0;1]^2\to[0;1]^2$ with no computable fixed point. • The Mandelbrot Set is computable, subject to the hyperbolicity conjecture.	There is a C^1 $f:[-1;1]^2\to[-1;1]$ polynom.-time computable s.t. solution $z:[0;1]\to[-1;1]$ to ODE $\dot{z}=f(z,t)$, $z(0)=0$ is not polynom.-time computable unless $\mathcal{P}=\mathcal{PSPACE}$. For polytime analytic *f*, *z* is also polytime.	• Computational Benefit of Smoothness: A Rigorous Parameterized Complexity Analysis of Operators on Gevrey's Hierarchy • Unform Complexity of Operators on Compact Sets	•The Lorenz Attractor exists. •The Kepler Conjecture is true •Billions of digits of π... •Numerical verification of the Riemann Hypothesis	•Coil design of Wendelstein 7-X •COSTAR for Hubble Space Telescope •Sleipner A offshore platform design
Type of Problems; Specification	Computability of certain real numbers, multi-/ functions etc., formalized and precisely	Complexity of real functions and subsets; formal and precise	Natural parameters + optimal explic. algorithms computing multi-/functions & operators	certain real constants; formal and precise	(individual) design-/ parameters; informal
Methods	prove existence / impossibility of algorithm solving some problem	existence / impossibility of polynomial-time algorithm	devise new & rigorously analyze existing algorithms	case-by-case	heuristical recipes and experience
Intermediate precision	unbounded	polynomially bounded	polynomially bounded	linear growth	fixed: hardware `float/double`
Programming	Type-2 Machine / real PCF	Oracle Turing machine	`Java / iRRAM + MPFR`	`GMP` + individually taylored software	`NAG, MATLAB`
Consistent semantics?	yes, closed under composition	yes, closed under composition	yes, multivalued tests → closed under composition	custom / ad-hoc	↯
Notion of Efficiency	↯ (recursive enumerability)	time polynomial in the binary output precision	time polynomial in a (fine-grained choice of) parameters	quadratic or softly linear time	absolute time (constant factors)
Correctness/ Optimality Demonstration	formal proof, derived from specification	formal proof + quantitative analysis	formal proof, quantitative analysis, adversary argument	analytic proof, extensive checking, 2-way calculations	empirical evaluation / benchmark suites
Establishing optimality	embedding undecidable (e.g. halting) problems, Weihrauch-reduction	embedding discrete (e.g. $\mathcal{NP}\neq\mathcal{P}/\mathcal{PSPACE}$) hard problems into continous	adversary arguments from IBC, adapted to bit model; (2nd order) polytime reduction	irrelevant	competition with other methods on said benchmarks

Benefits of Complexity Theory over Continuous Universes

iRRAM

- Full specification (input/output behavior)
- of problems over real numbers, functions, sets
- Consistent semantics (e.g. tests) closed under composition for modular approach to software
- Canonical interface declarations (TTE+2nd ord.)
- Rigorous convergence & runtime analyses
- turning recipes and heuristics into algorithms
- Realistic performance guarantees in bit model
- and optimality proofs from IBC or relative to common conjectures in classical complexity
- and parametrized for fine-grained predictions

Computable Real Numbers

Call $r\in\mathbb{R}$ computable if

Theorem: For $r\in\mathbb{R}$, the following are equivalent:

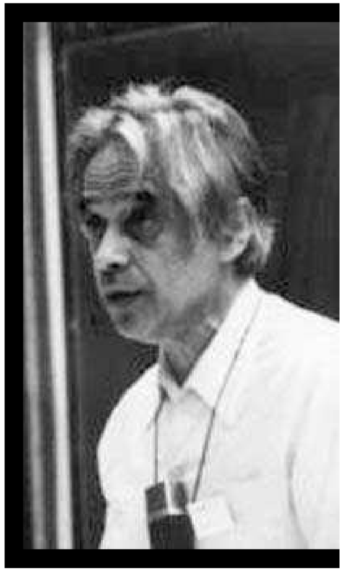
0.0111111...=0.100000...
0.010101... + 0.101010...=?

a) r has some/all binary expansion decidable

b) There is an algorithm printing, on input $m\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r-a/2^m|\leq 2^{-m}$.

$\Leftrightarrow r\in [q_n\pm\varepsilon_n]$

c) There is an algorithm printing two sequences $(q_n)\subseteq\mathbb{Q}$ and (ε_n) with $|r-q_n|\leq\varepsilon_n\to 0$



numerators+ denominators

b) $\Leftrightarrow$ c) holds *uniformly*, $\Leftrightarrow$ a) does not [Turing'37]

interval arithmetic

Ernst Specker (1949): (c) $\Leftrightarrow$ *Halting problem* plus (d)

d) There is an algorithm printing $(q_n)\subseteq\mathbb{Q}$ with $q_n\to r$.

$H := \{ \langle B,\underline{x}\rangle : \text{algorithm } B \text{ terminates on input } \underline{x} \} \subseteq \mathbb{N}$

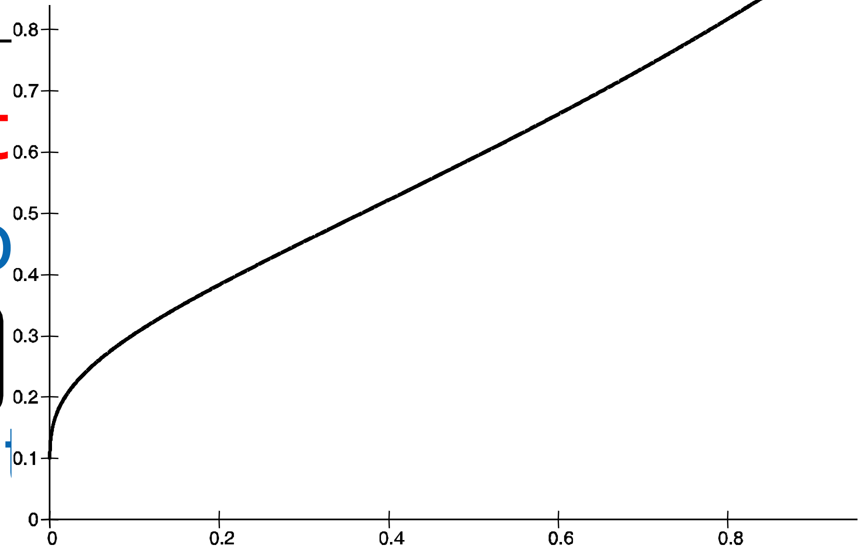
Real Function Complexity

Function $f:[0,1]\to\mathbb{R}$ comput

if some TM can, on input o

$(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|<2^{-m}$

in time $t(n)$ output $b\in\mathbb{Z}$ wi



Examples: a) $+, \times, \exp$ polytime on [0;1]!

b) $f(x)\equiv\sum_{n\in L} 4^{-n}$ iff $L\subseteq\{0,1\}^*$ polytime-decidable

c) sign, Heaviside not computable

$1/\ln(e/x)$ not polytime-computable

Lipschitz, Hölder

$|x-y|<2^{-\mu(n)} \Rightarrow |f(x)-f(y)|<2^{-n}$

Observation i) f computable $\Rightarrow$ continuous.

ii) If f is computable in time $t(n)$, then $t(n)$ is a modulus of uniform continuity of f.

Three Effects in Real Computation

that numerical scientists might be interested in / should be aware of

cmp. "feasible real-RAM"

a) natural emergence of multivaluedness (aka non-extensionality) → semantics of "$<_n$"

b) Uniform computation may require *discrete advice*

Example: Given $x\in\mathbb{R}$, compute $n\in\mathbb{Z}$ with $n\geq x$.

Example (E.Specker'69): Given $a_0,\ldots,a_{d-1}\in\mathbb{C}$ compute a d-tuple $z_1,\ldots,z_d\in\mathbb{C}$ of roots of $p(z):=a_0+z\cdot d_1+\ldots+z^{d-1}\cdot a_{d-1}+z^d$ with multiplicities.

Finding an eigenvector basis to a given real

is *un*computable;

Example: $+$, $\exp$ computable in time polynomial in n on $[0;1]$; on $[0;2^k]$: $+$ in time polynomial in $n+k$, $\exp$ in time polynomial in $n+2^k$.

independent of x on *compact* dom

Recap on Structural Complexity

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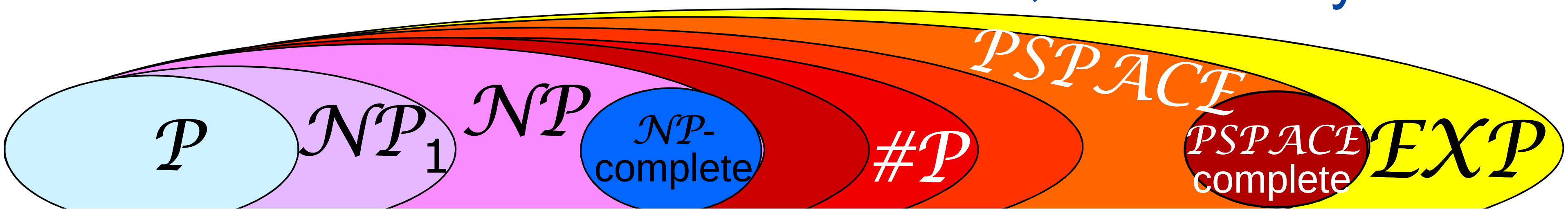
- $L \subseteq \{0,1\}^*$ polynomial-time decidable if a Turing Machine can, given $\underline{v} \in \{0,1\}^n$, tell whether $\underline{v} \in L$ or $\underline{v} \notin L$ within a number of steps polynomial in $n = |\underline{v}|$. $\mathcal{P}$
- $f: \{0,1\}^* \to \{0,1\}^*$ polynomial-time computable if… $\mathcal{FP}$
- L is polynomial-space decidable if can tell whether $\underline{v} \in L$ or $\underline{v} \notin L$ using at most polynomially many memory bits. $\mathcal{PSPACE}$
- $L = \{\ \underline{v} \in \{\ \mathbf{1}\ \}^* : \exists \underline{w} \in \{0,1\}^{p(|\underline{x}|)} : \langle \underline{v}, \underline{w} \rangle \in V\ \}$ with $p \in \mathbb{N}[N]$ and $V \in \mathcal{P}$ is called polynomial-time verifiable. $\mathcal{NP}$ $\mathcal{NP}_1$
- $f \in \mathcal{FP}$ is a polynomial-time reduction from A to B ("$A \preccurlyeq^{\mathrm{P}} B$") if it holds: $\forall \underline{v}: \underline{v} \in A \Leftrightarrow f(\underline{v}) \in B$.

$B \in \mathcal{NP}$ is $\mathcal{NP}$-complete if $A \preccurlyeq^{\mathrm{P}} B$ for every $A \in \mathcal{NP}$.



Nonuniform Complexity of Operators

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f:[0;1]→[0;1] polytime computable (⇒ continuous)

- Max: $f \to \mathrm{Max}(f): x \to \max\{ f(t): t \leq x \}$

  Max(f) computable in exp. time;

  polytime-computable **iff** $\mathcal{P}=\mathcal{NP}$

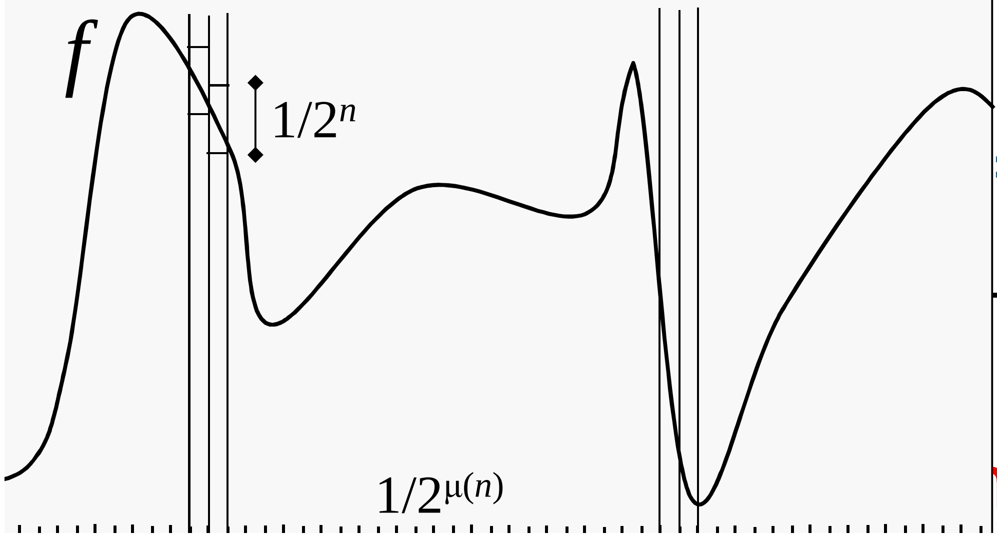
even when restricting to $f \in C^\infty$

$\{ \langle \mathrm{bin}(a),\mathrm{bin}(b),2^n \rangle : \exists w \leq a \leq 2^m, f(w/2^m) >_n b/2^n, m:=\mu(n) \}$

- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$

Finding starting points for Newton Iteration is hard

μ polyn. modulus of continuity



ntial time;

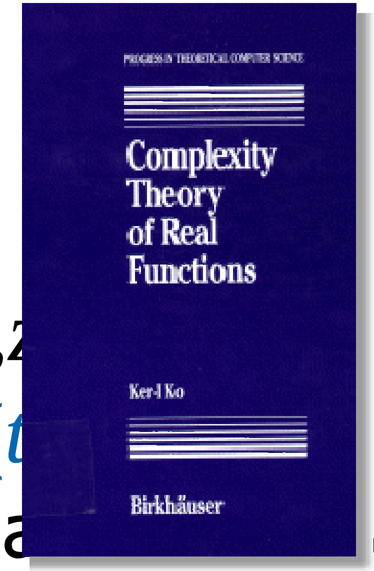
$\mathcal{FP}=\#\mathcal{P}$

$=\#\mathcal{P}_1$

$\to z$: $\dot{z}(t)=f(t,z$

le solution $z(t$

nplete" [Kawa… al]



[Friedman&Ko'82ff]

'Max is $\mathcal{NP}$-hard'

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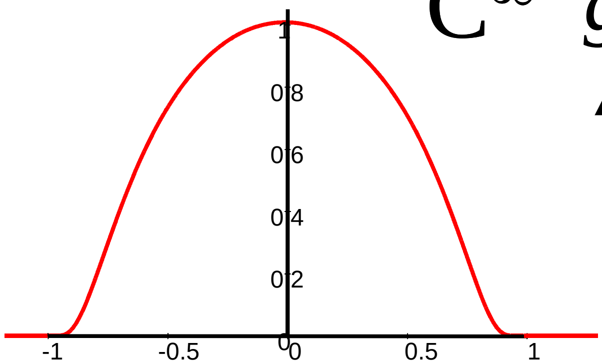
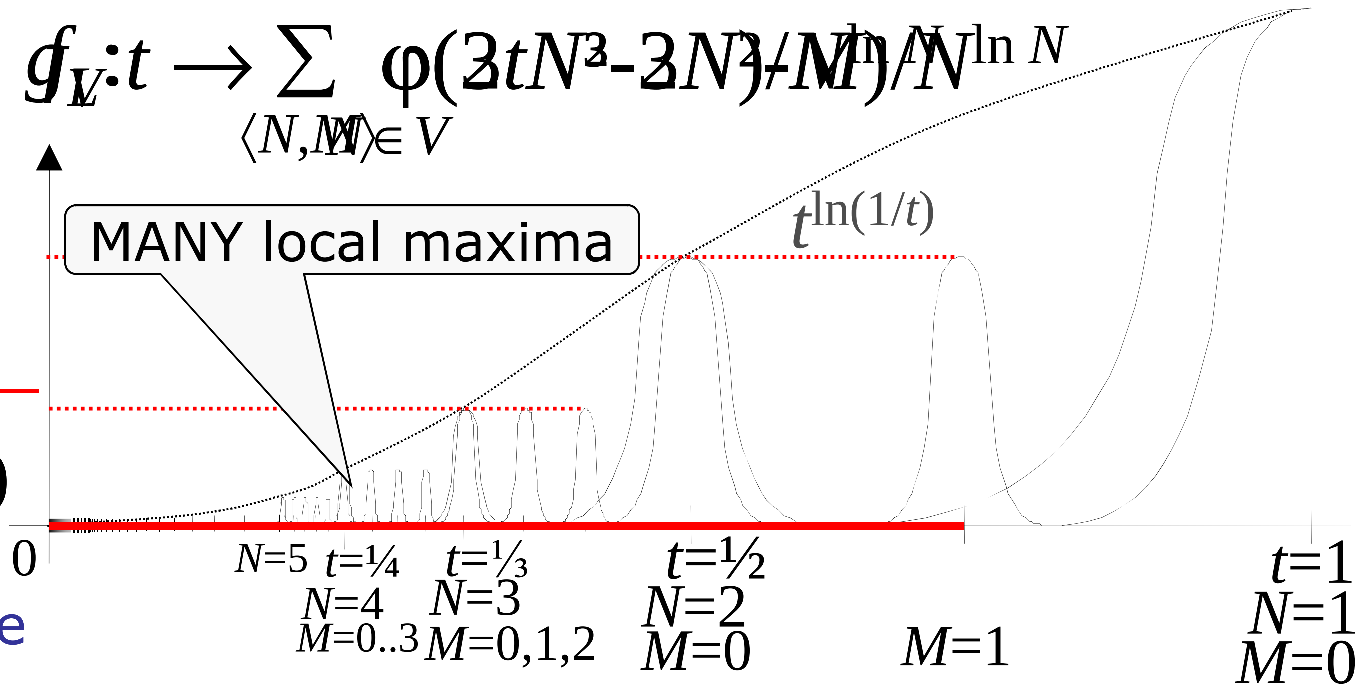
$\mathcal{NP} \ni L = \{\, N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle \in V \,\}$, $V\in\mathcal{P}$ polytime

$C^\infty \quad g_V: t \to \sum_{\langle N,M\rangle\in V} \varphi(2tN^2-2N^2-M)/N^{\ln N}$



$\varphi(t) = \exp(-t^2/1-t^2)$

C^∞ 'pulse' function
polytime computable



To every $L\in\mathcal{NP}$ there exists a polytime computable C^∞ function $g_L:[0,1]\to\mathbb{R}$ s.t.:
$[0,1]\ni t\to\max g_L|_{[0,t]}$ again polytime iff $L\in\mathcal{P}$

Complexity of PDEs

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- [Pour-El&Richards'81, ..., Weihrauch&Zhong'02] In-/computability of the *Wave Equation* (hyperbolic)
- Computability of some non-/linear PDEs: [Weihrauch&Zhong'99ff, Brattka&Yoshikawa'06, ...]

Poisson's Equation: $\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega$

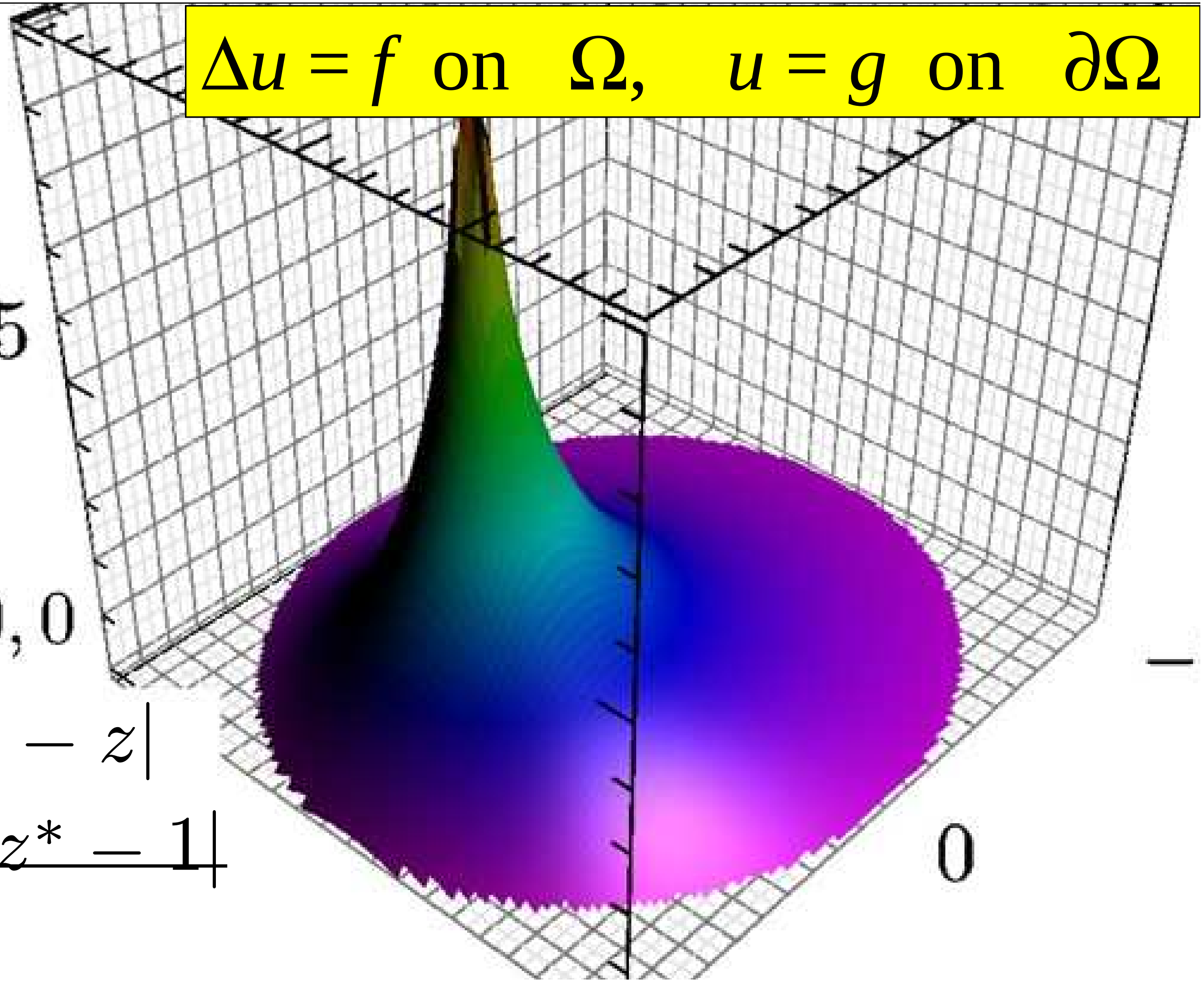
- electrostatic / gravitational potential of the charge/ mass distribution f with boundary condition g
- 2nd order, linear, elliptic: homog. $(f,0)$ and inh. $(0,g)$
- 'fundamental' solutions $\ln|\underline{x}|$ (2D) and $1/|\underline{x}|$ (3D)
- 'explicit' Green's functions for various domains,
- solution formula on the complex unit disc; e.g. $g=0$

$$u(z) = -\frac{1}{2\pi} \cdot \int_{|w|=1} \ln \frac{|w-z|}{|w \cdot z^* - 1|} \cdot f(w)\, dA(w)$$

Green's Function in 2D

$\Delta u = f$ on Ω, $u = g$ on $\partial\Omega$

$G(1,w)$



$$G(z,w) = \ln \frac{|w-z|}{|w \cdot z^* - 1|}$$

Complexity of $\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega$

Theorem: B_d := closed d-dim. Euklidean unit ball

a) For every polytime $f:B_d \to \mathbb{R}$ and $g:\partial B_d \to \mathbb{R}$, there exists a unique C^2 solution u and u is computable in exponential time.

b) If $\mathcal{FP}=\#\mathcal{P}$, then u is even polytime computable.

c) There exists a polytime $f \in C^\infty$ such that
u to $g \equiv 0$ is polytime iff $\mathcal{FP}=\#\mathcal{P}$.

d) For $d>1$ there is a polytime $g \in C^\infty$ s.t.
u to $f \equiv 0$ is polytime iff $\mathcal{FP}=\#\mathcal{P}$.

[Kawamura, Steinberg, Z.'14]

$$u(z) = -\frac{1}{2\pi} \cdot \int_{|w|=1} \ln \frac{|w-z|}{|w \cdot z^* - 1|} \cdot f(w)\, dA(w)$$

Complexity of Operators (cont.d)

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f:[0;1]→[0;1] polytime computable + C^ω (analytic)

- Max: $f \to \mathrm{Max}(f): x \to \max\{ f(t): t \leq x \}$
  polytime-computable !
- $\int: f \to \int f: (x \to \int_0^x f(t)\, dt)$
  polytime-computable !
- dsolve: $C[0;1]\times[-1;1] \ni f \to z: \quad \dot{z}(t)=f(t,z), \ z(0)=0.$
  polytime-computable !

[N.Müller'87+'95, Müller&Moiske'93, Bournez et al.'11ff]

Claims are *non*-uniform: parameterized

- Runtime (polyn. degree) of computing $\mathrm{Max}(f)$ in dependence on runtime (polyn.degree) of f ?
- Discrete advice about (in addition to apprx. to) f employed by computation of $\mathrm{Max}(f)$?

Representing Power Series on the closed unit disc

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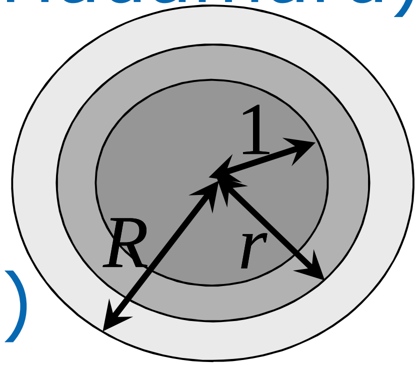
$\sum_j c_j \cdot z^j$

incomputable [ZhWe'01]

- radius of convergence $R=1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $k\in\mathbb{N}$: $|c_j|\leq 2^k/r^j$ (Cauchy-Hadamard)
- $\mathbb{N} \ni k :\geq 1/\log(r) = \Theta(1/(r-1))$
- tail bound $|\sum_{j\geq N} c_j z^j| \leq C\cdot(|z|/r)^N/(1-|z|/r)$

discrete advice



Complexity uniform in $|z|\leq 1$: (i.e. $R>1$)

Convergence degrades es as $r\rightarrow 1$; quantitatively?

output precision

parametrized running time

Theorem: Represent series $\sum_j c_j \cdot z^j$ with $R>1$ as $(a_{jm})\subseteq\mathbb{Z}$ s.t. $|c_j-a_{jm}/2^m|<2^{-m}$ ($\equiv^{\mathrm{P}}\rho^\omega$) and $k\in\mathbb{N}$ as above

The following are uniformly computable in time polyn. in $n+k$: i) eval, ii) sum, iii) product, iv) derivative, v) anti-derivative $\int$, vi) Max

[Kawamura, Müller, Rösnick, Z.'14]

Representing Continuous Functions

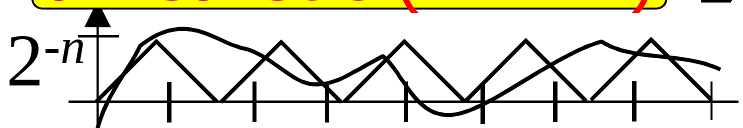
Cantor

TTE: Represent space Z via surj. $\delta:\subseteq\{0,1\}^\omega\to Z$

On <u>compact</u> $\mathcal{Lip}_1 := \{\ f:[0;1]\to[0;1]$ 1-Lipschitz $\}$, representing f via a ρ^ω-name of $(f(k/2^n))_{\langle k,n\rangle}$ renders operator Apply: $\mathcal{Lip}_1\times[0;1] \ni (f,x)\to f(x)$ computable in time $O(1)^n$, not in *sub*exp. time for <u>any</u> representat.

unrealistic (iRRAM) 2^{-3}

$\geq 2^{2^{n-1}}$ functions pairwise differing when evaluating up to error 2^{-n} but only $2^{t(n)}$ different initial segments of δ-names that can be read within $t(n)$ steps. q.e.d.



$\approx 2^{n-1}$ 'hats' / C^∞ pulses, $2^{2^{n-1}}$ combinations

sequential access!

$\mathcal{Lip}_1$ has exponential metric entropy [Weihrauch'03]

IBC: exact (or of fixed precision) *unit cost* queries

Friedman&Ko: <u>oracle access</u> to f via approx. eval.:

Given $q\in\mathbb{D}_n:=\{a/2^n: a\in\mathbb{Z}\}$ return $p\in\mathbb{D}_n$: $|f(q)-p|<2^{-n}$.

Representing Continuous Functions

Cantor

TTE: Represent space Z via surj. $\delta :\subseteq \{0,1\}^{\omega} \to Z$

Definition [Kawamura&Cook'10]:

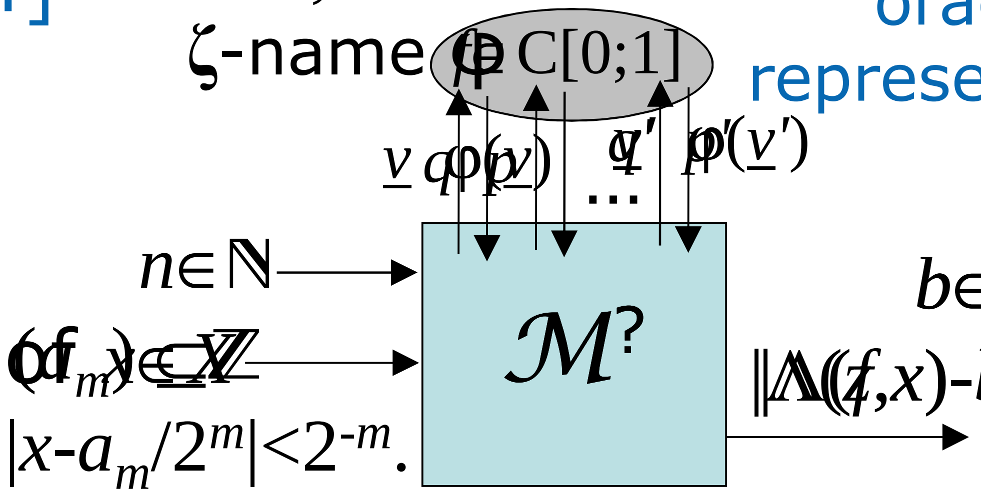
A 2nd-order representation is a surj. $\boldsymbol{\delta} :\subseteq \{0,1\}^{**} \to Z$

where $\{0,1\}^{**} := (\{0,1\}^{*})^{(\{0,1\})^{*}}$ denotes Baire Space

[Kawamura&Pauly'14]

Example: Define a $\rho^{\mathbb{D}}$-name of $f \in C[0;1]$ as any φ s.t. $\left| f(\,\mathrm{bin}(\underline{v})/2^{|\underline{v}|}\,) - \mathrm{bin}(\varphi(\underline{v}))/2^{|\underline{v}|} \right| < 2^{-|\underline{v}|}$

$z \in Z$, ζ-name φ



oracle φ represents f

Oracle machine computing a functional $\Lambda :\subseteq C[0;1] \times \mathbb{R} \to \mathbb{R}$

Friedman&Ko: oracle access to f via approx. eval.:

Given $q \in \mathbb{D}_n := \{a/2^n : a \in \mathbb{Z}\}$ return $p \in \mathbb{D}_n$: $|f(q)-p| < 2^{-n}$.

Representing Real Analytic Functions

Definition: $C^\omega[-1,1] := \{\, f{:}[-1;1]\to\mathbb{R}$ restriction of complex differentiable $g{:}U\to\mathbb{C}$, $[-1;1]\subseteq U\subseteq\mathbb{C}$ open $\}$

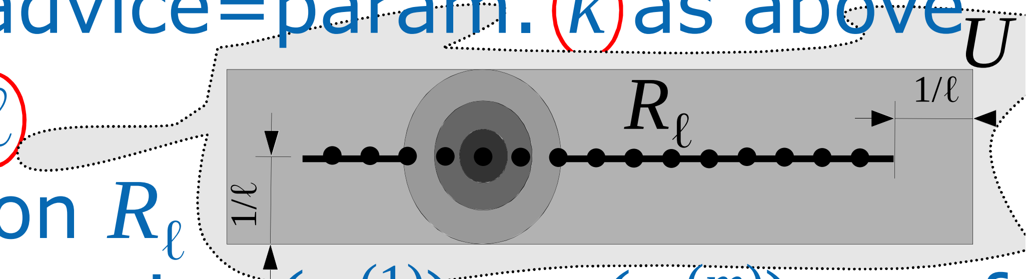
Fact: $f\in C^\omega[-1;1] \Leftrightarrow$ $f\in C^\infty \wedge \exists k\in\mathbb{N}\ \ \forall j{:}\ \ \|f^{(j)}\| \leq 2^k\cdot k^j\cdot j!$

$\Leftrightarrow f =$ finitely many local power series on $[-1;1]$

Theorem: The following (2nd-ord) representations of $C^\omega[-1,1]$ are parameterized poly-time equivalent:

a) a $\rho^{\mathbb{D}}$-name of f with advice=param. k as above

b) a $\rho^{\mathbb{D}}$-name of f with ℓ s.t. $R_\ell\subseteq U$ and $|g|\leq 2^\ell$ on R_ℓ



c) ρ^ω-names of m power series $(c_j^{(1)}),\ldots,(c_j^{(m)})\subseteq\mathbb{R}$ of f around equidistant centers$\in[-1;1]$ s.t. $|c_j|\leq 2^m\cdot(2m)^j$

Friedman&Ko: A $\rho^{\mathbb{D}}$-name φ of $f\in C[-1;1]$ maps every $q\in[-1;1]\cap\mathbb{D}_n$ to some $p\in\mathbb{D}_n$: $|f(q)-p|\leq 2^{-n}$

Theorem: On $C^\omega[0,1]$, i) eval ii) sum … vi) max are computable within parameterized polyn. time

Gevrey's Function Hierarchy

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Definition (Maurice Gevrey 1918, studying PDEs)

$g \in \mathcal{G}^{\ell}_{k}[-1;1] :\Leftrightarrow \ \forall j: \|g^{(j)}\| \leq 2^k \cdot k^j \cdot j^{j \cdot \ell}$ $\qquad \mathcal{G}^{\ell} := \bigcup_k \mathcal{G}^{\ell}_k$

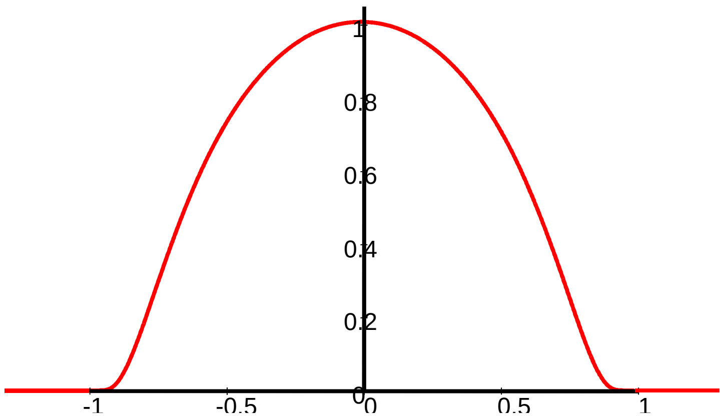
Fact: $f \in C^{\omega}[-1;1] \Leftrightarrow f \in C^{\infty} \wedge \exists k \in \mathbb{N} \ \ \forall j: \ \|f^{(j)}\| \leq 2^k \cdot k^j \cdot j!$

$\mathcal{G}^1 = C^{\omega}$

Example: The following g is not analytic but in $\mathcal{G}^3[-1;1]$

$$g(x) := \exp\left(\frac{x^2}{x^2-1}\right) \text{ for } |x| \leq 1,$$

$$g(x) := 0 \qquad \text{ for } |x| \geq 1$$



Komplexity on Gevrey's Hierarchy

Definition (Maurice Gevrey 1918, studying PDEs)

$g\in \mathcal{G}^{\ell}_{k}[-1;1] :\Leftrightarrow \forall j: \|g^{(j)}\| \le 2^{k}\cdot k^{j}\cdot j^{j\cdot\ell}$ $\qquad \mathcal{G}^{\ell} := \bigcup_{k} \mathcal{G}^{\ell}_{k}$

Labhalla&Lombardi&Moutai 2001:

$\Rightarrow \exists B\ \forall n\ \exists p\in \mathbb{D}[X]: \deg(p)<B\cdot n^{\ell}\ \ \|g-p\|\le 2^{-n} \ \Rightarrow g\in \mathcal{G}^{2\ell-1}[-1;1]$

Theorem [Kawamura,Müller,Rösnick,Z.'14]:
The following (2nd-ord) representations
of $\mathcal{G}[-1;1] := \bigcup_{\ell} \mathcal{G}^{\ell}[-1;1]$ are computationally
equivalent up to time polynomial in $(k+n)^{\ell}$:
a) $\rho^{\mathbb{D}}$-name of f with advice k,ℓ as above
b) sequence $p_n\in \mathbb{D}[X]$ with $\deg(p_n)<B\cdot n^{\ell}\ \ \|g-p_n\|_\infty \le 2^{-n}$.
Moreover they render the following operations
computable in time polynomial in $(k+n)^{\ell}$:
i) eval, ii) sum, iii) product, iv) d/dx, v) $\int$, vi) max
W.r.t. $\rho^{\mathbb{D}}$, max (vi) on $\mathcal{G}^{\ell}_{1}$ requires time $\Omega(n^{\ell})$.

Representing Compact Euclidean Subsets

Fix closed $S\subseteq[0;1]^d$ and $1\leq p\leq\infty$.

ψ_p-name φ: $\varphi\langle \underline{q}\rangle=$**1** for $\underline{q}\in\mathbb{D}_n{}^d$ with $\mathrm{Ball}_p(\underline{q},2^{-n})\cap S\neq\emptyset$

$\varphi\langle \underline{q}\rangle=$**0** for $\underline{q}\in\mathbb{D}_n{}^d$ with $\mathrm{Ball}_p(\underline{q},2^{-n+1})\cap S=\emptyset$

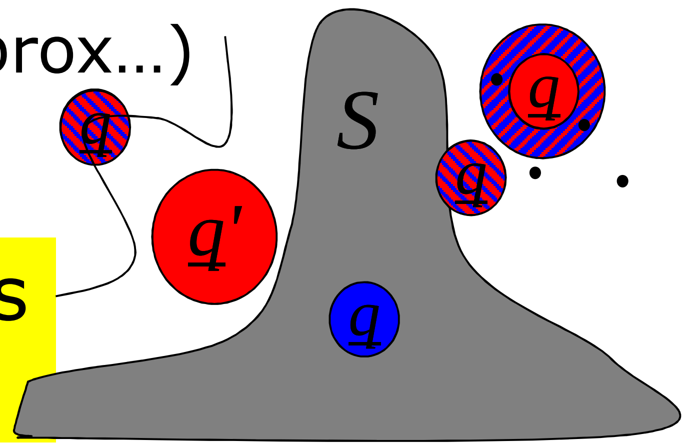
δ_p-name: $\rho^{\mathbb{D}}$-name of $\mathrm{dist}_{S,p}:[0;1]^d\ni\underline{x}\to\min\{|\underline{x}-\underline{s}|_p:\underline{s}\in S$

ω_p-name φ: $\varphi\langle \underline{q}\rangle=$**1** for $\underline{q}\in\mathbb{D}_n{}^d$ with $\mathrm{Ball}_p(\underline{q},2^{-n})\subseteq S$

$\varphi\langle \underline{q}\rangle=$**0** for $\underline{q}\in\mathbb{D}_n{}^d$ with $\mathrm{Ball}_p(\underline{q},2^{-n})\cap S=\emptyset$

⋮

(symm./rel. distance, multiv. best approx…)



A 2nd-order representation of Z is a surjective $\boldsymbol{\delta}:\subseteq(\{0,1\}^*)^{(\{0,1\})^*}\to Z$

[Brattka&Weihrauch'99, Z.'02+'04, Braverman'04, Zhao&Müller'08, Rösnick'14]

Representing Compact Euclidean Subsets

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ω_p-name

Fact: $\psi\equiv\delta$, $\equiv\omega$ on regular sets [Bra&Wei'99,Z.'02]

Theorem [Brav.'04,Zh&Mü'08,Rösn.'14]:

unless $\mathcal{P}=\mathcal{NP}$

a) It holds $\boldsymbol{\delta}\preccurlyeq^{\mathrm{P}}\boldsymbol{\psi}\preccurlyeq^{\mathrm{P}}\boldsymbol{\omega}$. Projection is $\mathcal{NP}$-"complete

b) There is a S with $\mathrm{dist}_{S,1}$ polytime but <u>not</u> $\mathrm{dist}_{S,\infty}$

c) For any (!) fixed $1\leq p,p'\leq\infty$ it holds $\boldsymbol{\psi}_p\equiv^{\mathrm{P}}\boldsymbol{\psi}_{p'}$

d) For <u>convex</u> sets of inner diameter $\geq 2^{-k}$, all three representations parametriz. polyn.time equivalent.

2^{nd}-Order Complexity Theory

TECHNISCHE UNIVERSITÄT DARMSTADT, Martin Ziegler

Function arguments $f\in C[0;1]$: 2^{nd}-order represent.

Apply: $\mathcal{Lip}_{2^\ell}[0;1]\times[0;1] \ni (f,x)\to f(x)$ computable in

parameterized time polynomial in $n+\ell$.

Observation: If f is computable in time $t(n)$, then $t(n)$ is a modulus of uniform continuity of f.

Apply: $C[0;1]\times[0;1] \ni (f,x)\to f(x)$ requires time

depending on a modulus μ of continuity:

"Parameter" μ is not ℕ-valued but $\mathbb{N}^\mathbb{N}$-valued!

Example: $\lambda^3\big(\lambda(n^2)\cdot n+\lambda^2(n)\big)+n^{17}$

Def [Mehlhorn'76]**:** A 2^{nd}-order polynomial $P(n,\lambda)$ is a term over $+$, $\times$, integer constants and (1^{st}-order) variable n (ranging over ℕ) and 2^{nd}-order variable λ (ranging over $\mathbb{N}^\mathbb{N}$)

2^{nd}-Order Polynomials

Observation: a) 2^{nd}-order polynomials are closed under *both* kinds of composition

$(Q \circ P)(n,\lambda) := Q\big(P(n,\lambda),\lambda\big)$ and $(Q \square P)(n,\lambda) := Q\big(n,P(\cdot,\lambda)\big)$

b) For $\lambda \in \mathbb{N}[n]$, $P(n,\lambda)$ is an ordinary polynomial.

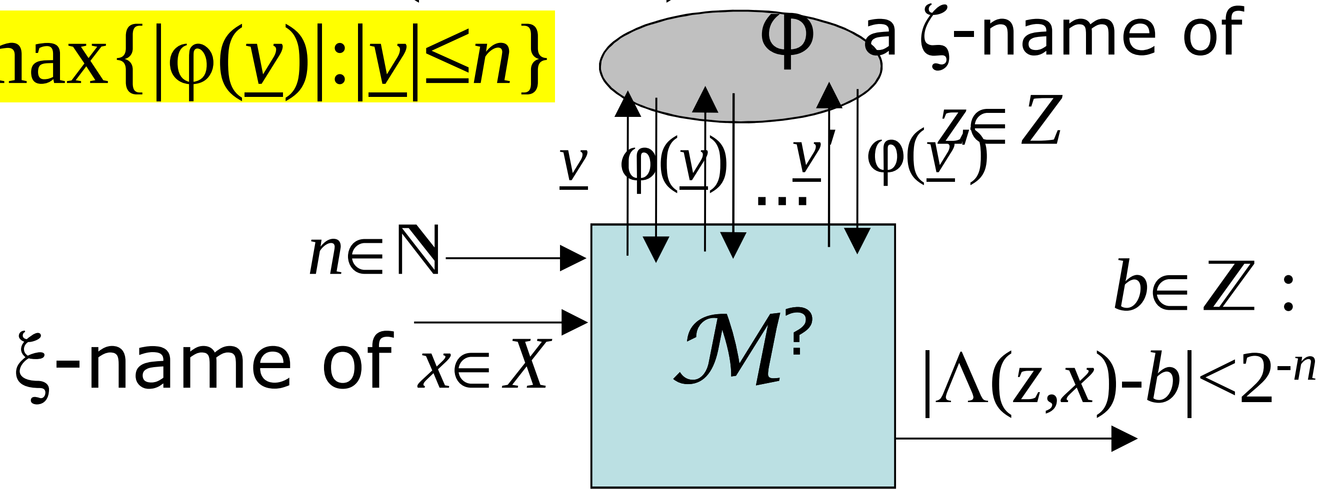
Example: $\lambda^3\big(\lambda(n^2)\cdot n+\lambda^2(n)\big)+n^{17}$

Def [Mehlhorn'76]**:** A 2^{nd}-order polynomial $P(n,\lambda)$ is a term over $+$, $\times$, integer constants and (1^{st}-order) variable n (ranging over $\mathbb{N}$) and 2^{nd}-order variable λ (ranging over $\mathbb{N}^{\mathbb{N}}$)

2^{nd}-Order Polyn.time Computation

A 2^{nd}-order representation is a surj. $\delta :\subseteq \{0,1\}^{**} \to Z$ with Baire Space $\{0,1\}^{**} := (\{0,1\}^*)^{(\{0,1\})^*}$ and grading $|\varphi|(n) := \max\{|\varphi(\underline{v})| : |\underline{v}| \le n\}$

A $\rho^{\mathbb{D}}$-name of $f \in C[0;1]$ is any φ s.t.
$\left| f(\operatorname{bin}(\underline{v})/2^{|\underline{v}|}) - \operatorname{bin}(\varphi(\underline{v}))/2^{|\underline{v}|} \right| < 2^{-|\underline{v}|}$



Oracle machine computing a functional $\Lambda :\subseteq Z \times X \to \mathbb{R}$

Def [Kapron&Cook'96]: $\mathcal{M}^?$ runs in 2^{nd}-order poly. time if #steps $\le P(n,|\varphi|)$ for every x and ζ-name φ

Def [Mehlhorn'76]**:** A 2^{nd}-order polynomial $P(n,\lambda)$ is a term over $+$, $\times$, integer constants and (1^{st}-order) variable n (ranging over $\mathbb{N}$) and 2^{nd}-order variable λ (ranging over $\mathbb{N}^{\mathbb{N}}$)

iRRAM

- Full specification (input/output behavior)
- of problems over real numbers, functions, sets
- Consistent semantics (e.g. tests) closed under composition for modular approach to software
- Canonical interface declarations (TTE+2nd ord.)
- Rigorous convergence & runtime analyses
- turning recipes and heuristics into algorithms
- Realistic performance guarantees in bit model
- and optimality proofs from IBC or relative to common conjectures in classical complexity
- and parametrized for fine-grained predictions